MATH2060B Mathematical Analysis II, Test 2

Answer ALL Questions 20 Mar 2019, 10:30-11:15

1. (i) (8 *ponits*) State and show the Mean Value Theorem for Integrals.
   (See the note.)

   (ii) (8 *ponits*) Show that $\lim_{n\to\infty} \int_0^{\frac{\pi}{2}} \sin^n x \cos^n x dx = 0$.
   Notice that since $\sin^n x \cos^n(x) = (\frac{1}{2}\sin 2x)^n$, we have $|\int_0^{\frac{\pi}{2}} \sin^n x \cos^n x dx| \leq \frac{1}{2^n}(\pi/2) \to 0$.

   (iii) (8 *ponits*) Show that the integral $\int_1^\infty \exp(-x^2)dx$ is convergent.
   The result follows from $\exp(-x^2) \leq \exp(-x)$ for all $x \geq 1$ at once.

   (iv) (8 *ponits*) Show that if g is any bounded continuous function defined on $[1, \infty)$, then the integral $\int_1^\infty \exp(-x^2)g(x)dx$ is convergent.
   Suppose that $|g(x)| \leq M$ for some $M > 0$. Then

$$|\int_A^{A'} \exp(-x^2)g(x)dx| \leq M \int_A^{A'} \exp(-x^2)dx$$

   for all $A' > A > 1$. So, the result follows from Part (iii) and the Cauchy Theorem immediately.

2. (8 *ponits*) Define a function $f : [0,1] \longrightarrow \mathbb{R}$ by $f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{otherwise.} \end{cases}$

   Find the upper integral $\overline{\int_0^1} f(x)dx$ and the lower integral $\underline{\int_0^1} f(x)dx$.
   Answer: Let $P : 0 = x_0 < \cdots x_n := 1$ be any partition on $[0,1]$.
   Note that we always have $m_i(f,P) := \inf\{f(x) : x \in [x_{i-1}, x_i]\} = 0$ for all $i = 1, ..., n$ because $[x_{i-1}, x_i] \cap \mathbb{Q}^c \neq \emptyset$. So, we have $\underline{\int_0^1} f(x)dx = 0$.
   On the other hand, notice that $M_i(f,P) := \sup\{f(x) : x \in [x_{i-1}, x_i]\} = x_i^2$ for all $i = 1, .., n$. This implies that if we let $g(x) := x^2$ for $x \in [0,1]$, then we have $M_i(f,P) = M_i(g,P)$ for all i and for all partitions P. This gives $\overline{\int_0^1} f(x)dx = \overline{\int_0^1} g(x)dx$. Since g is continuous, we have $\overline{\int_0^1} g(x)dx = \int_0^1 x^2 dx = 1/3$ by using the Fundamental Theorem of Calculus.

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